A Semiclassical Birkhoff Normal Form

Léo Morin

Inter'Action - 21 Mai 2019

UNIVERSITÉ DE RENNES 1

1. Semiclassical Dynamics

2. The Birkhoff Normal Form

Classical dynamics

Classical Hamiltonian (Energy):

$$H(x,\xi) = \xi^2 + V(x), \quad x,\xi \in \mathbf{R}.$$

Classical dynamics

Classical Hamiltonian (Energy):

$$H(x,\xi) = \xi^2 + V(x), \quad x,\xi \in \mathbf{R}.$$

Classical dynamics:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \frac{\partial H}{\partial \xi}(x(t),\xi(t))$$
$$\frac{\mathrm{d}\xi}{\mathrm{d}t}(t) = -\frac{\partial H}{\partial x}(x(t),\xi(t)).$$

Classical dynamics

Classical Hamiltonian (Energy):

$$H(x,\xi) = \xi^2 + V(x), \quad x, \xi \in \mathbf{R}.$$

Classical dynamics:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t}(t) &= \frac{\partial H}{\partial \xi}(x(t), \xi(t)) \\ \frac{\mathrm{d}\xi}{\mathrm{d}t}(t) &= -\frac{\partial H}{\partial x}(x(t), \xi(t)). \end{aligned}$$

Then H is constant along the solution $(x(t), \xi(t))$:

$$H(x(t), \xi(t)) = H(x_0, \xi_0).$$

Quantum dynamics

Quantum Hamiltonian:

$$\mathcal{L}_\hbar = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \quad \text{on } \mathsf{L}^2(\mathbf{R}).$$

Quantum dynamics

Quantum Hamiltonian:

$$\mathcal{L}_\hbar = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \quad \text{on } \mathsf{L}^2(\mathbf{R}).$$

Quantum dynamics:

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{L}_\hbar \psi.$$

Quantum dynamics

Quantum Hamiltonian:

$$\mathcal{L}_\hbar = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \quad \text{on } \mathsf{L}^2(\mathbf{R}).$$

Quantum dynamics:

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{L}_\hbar \psi.$$

The quantum energy is given by the eigenvalues of $\mathcal{L}_\hbar$:

$$0 < \lambda_1(\hbar) \leq \lambda_2(\hbar) \leq \ldots$$

Semiclassical dynamics

Link between the *classical* and *quantum* dynamics in the limit $\hbar \to 0$?

Semiclassical dynamics

Link between the *classical* and *quantum* dynamics in the limit $\hbar \to 0$?

There is a formula:

$$\mathcal{L}_\hbar = \mathrm{Op}_\hbar^w(H)$$

1 Semiclassical Dynamics

2 The Birkhoff Normal Form

Use the relation between H and $\mathcal{L}_\hbar$ to compute $\lambda_j(\hbar)$.

Use the relation between H and $\mathcal{L}_\hbar$ to compute $\lambda_j(\hbar)$.

$$\mathcal{U}_\hbar^* \mathcal{L}_\hbar \mathcal{U}_\hbar = \mathcal{N}_\hbar + \mathcal{R}_\hbar.$$

Harmonic oscillator

$$\mathcal{I}_\hbar = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$$

is given by:

$$\mathcal{I}_\hbar = \mathrm{Op}_\hbar^w(\xi^2 + x^2) = \mathrm{Op}_\hbar^w(|z|^2).$$

We know its spectrum:

$$\sigma(\mathcal{I}_\hbar) = \{\hbar, 3\hbar, 5\hbar, ...\}.$$

Theorem

There is a unitary operator $\mathcal{U}_\hbar$ *on* $\mathsf{L}^2(\mathbb{R})$ *such that:*

1. $\mathcal{U}_\hbar^* \mathcal{L}_\hbar \mathcal{U}_\hbar = \mathcal{N}_\hbar + \mathcal{R}_\hbar$,
2. $[\mathcal{N}_\hbar, \mathcal{I}_\hbar] = 0$,
3. $\mathcal{R}_\hbar$ *is "a rest of order* $\hbar^\infty$*",*
4. $\mathcal{N}_\hbar = \mathrm{Op}_\hbar^w(N)$ *where the Taylor series of* N *is*

$$N = |z|^2 + \sum_{\alpha,\ell} c_{\alpha\ell} |z|^{2\alpha} \hbar^\ell.$$

Then, the eigenvalues of $\mathcal{N}_\hbar$ are given by replacing the harmonic oscillator $|z|^2$ by its eigenvalues $\hbar, 3\hbar, \dots$

$$\lambda_1(\mathcal{N}_\hbar) = \hbar + \sum_{\alpha,\ell} c_{\alpha\ell}\hbar^{\alpha+\ell},$$

$$\lambda_2(\mathcal{N}_\hbar) = 3\hbar + \sum_{\alpha,\ell} c_{\alpha\ell}3^\alpha\hbar^{\alpha+\ell}\dots$$

and we have

$$\lambda_j(\mathcal{L}_\hbar) = \lambda_j(\mathcal{N}_\hbar) + \mathcal{O}(\hbar^\infty).$$

Idea of the proof.
We see functions of (x, ξ) as functions of $(z, \bar{z})$.
We approximate functions by their Taylor series:

$$H = |z|^2 + \sum_{\alpha\alpha'\ell} d_{\alpha\alpha'\ell} z^\alpha \bar{z}^{\alpha'} \hbar^\ell.$$

$$N = |z|^2 + \sum_{\alpha,\ell} c_{\alpha\ell} |z|^{2\alpha} \hbar^\ell.$$

Properties of Taylor Series

1. $\deg(z^{\alpha}\bar{z}^{\alpha'}\hbar^{\ell}) = \alpha + \alpha' + 2\ell$
2. If τ has valuation k, we write $\tau \in \mathcal{O}_k$
3. If $\tau_j \in \mathcal{O}_{k_j}$,

$$\mathrm{Op}_{\hbar}^{w}(\tau_1)\mathrm{Op}_{\hbar}^{w}(\tau_2) = \mathrm{Op}_{\hbar}^{w}(\tau_1 \star \tau_2)$$

   with $\tau_1 \star \tau_2 \in \mathcal{O}_{k_1+k_2}$.
4. Moreover,

$$\frac{i}{\hbar}[\tau_1, \tau_2] \in \mathcal{O}_{k_1+k_2-2}.$$

We look for $\mathcal{U}_\hbar = e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(\tau)}$. If

$$H = |z|^2 + R_3 + \mathcal{O}_4,$$

we have:

$$e^{\frac{i}{\hbar}\mathrm{Op}_\hbar^w(\tau)}\mathrm{Op}_\hbar^w(H)e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(\tau)} = \mathrm{Op}_\hbar^w\left(|z|^2 + R_3 - \frac{i}{\hbar}[|z|^2, \tau] + \mathcal{O}_4\right)$$

$\rightarrow$ Can we find τ such that $\frac{i}{\hbar}[|z|^2, \tau]$ removes the crossed terms of R_3 ?

Can we find τ such that $\frac{i}{\hbar}[|z|^2, \tau]$ removes the crossed terms of R_3 ?
Denote $d_{\alpha\alpha'\ell} z^\alpha \bar{z}^{\alpha'} \hbar^\ell$ a crossed term ($\alpha \neq \alpha'$) of R_3. For any coefficient c we have

$$\frac{i}{\hbar}[|z|^2, cz^\alpha \bar{z}^{\alpha'} \hbar^\ell] = c(\alpha' - \alpha) z^\alpha \bar{z}^\alpha \hbar^\ell.$$

Since $\alpha \neq \alpha'$, we can choose

$$c_{\alpha\alpha'\ell} = \frac{d_{\alpha\alpha'\ell}}{\alpha' - \alpha}$$

Then

$$\tau = \sum_{\alpha\alpha'\ell} c_{\alpha\alpha'\ell} z^{\alpha} \bar{z}^{\alpha'} \hbar^{\ell}$$

removes the crossed terms of R_3.

Then

$$\tau = \sum_{\alpha\alpha'\ell} c_{\alpha\alpha'\ell} z^{\alpha} \bar{z}^{\alpha'} \hbar^{\ell}$$

removes the crossed terms of R_3. We iterate with

$$|z|^2 + K_3 + R_4 + \mathcal{O}_5,$$

where K_3 only has non-crossed terms.

Thank you

- L. Charles, S. Vu Ngoc, Spectral Asymptotics via the semiclassical Birkhoff Normal Form, *Duke Math. J.*, 143(3) : 463-511, 2008.
- N. Raymond, *Bound States of the Magnetic Schrödinger Operator*, EMS Tracts in Mathematics 27, 2017.
- M. Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics 138, 2012.