# Universality for random permutations

## Colloque Inter'Actions en Mathématiques 2019

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## KPZ class : GUE

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ \overline{x_{1,2}} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x_{1,n}} & \overline{x_{2,n}} & \cdots & x_{n,n} \end{pmatrix}$$

$X_{i,j}$ i.i.d $\mathcal{N}_{\mathbb{C}}(0,1)$ (for $i<j$) and $X_{i,i}$ i.i.d $\mathcal{N}_{\mathbb{R}}(0,1)$.

$$F_2(s) = \lim_{n\to\infty} \mathbb{P}\left( (\lambda_{\max} - \sqrt{2n})\sqrt{2}n^{1/6} \leq s \right),$$

$F_2$: CDF of the GUE Tracy-Widom distribution.

# Longest increasing subsequence

- $\mathfrak{S}_n$: symmetric group, (the group of permutations of $\{1,\ldots,n\}$).
- $(\sigma(i_1),\ldots,\sigma(i_k))$ increasing subsequence of $\sigma$ of length $k$ if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) < \cdots < \sigma(i_k)$.
- $\ell(\sigma)$: the length of the longest increasing subsequence of $\sigma$.
- For example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$

$\ell(\sigma) = 5.$

# KPZ : Longest common subsequence

- $(\sigma(i_1),\ldots,\sigma(i_k))$ subsequence of $\sigma$ of length $k$ if $i_1 < i_2 <,\ldots,< i_k$.
- $LCS(\sigma_1,\sigma_2)$ the length of the longest common subsequence of two permutations.
- For example:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$LCS(\sigma) = 2.$

# KPZ class : Last passage percolation

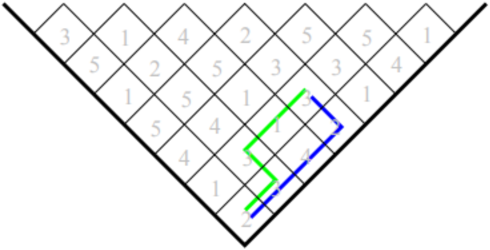

Figure: Last passage percolation (image of Borodin and Gorin (2012))

# KPZ class : Ballistic model

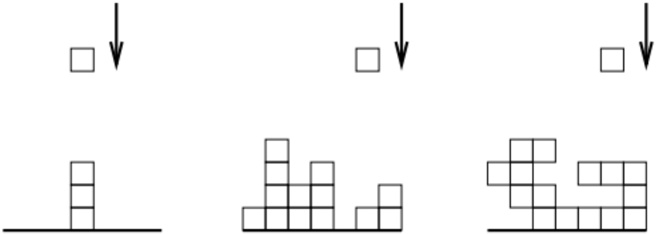

Figure: Ballistic model (image of Borodin and Gorin (2012))

# Universality

- Models :
  - Radom Matrices (GUE, Wigner etc.)
  - Interacting particles (ASEP, TASEP, push-ASEP etc.)
  - Growth models
  - Random permutations / random partitions
  - Random tiling
  - 6-vertex model
- Limiting objects / transitions :
  - Tracy-Widom distribution / Airy Kernel
  - Semi-circular law
  - (Discrete) Sine process
  - Baik-Ben Arous-Péché phase transition.

| | GUE + Other random matrices | Uniform permutation |
|---|---|---|
| Largest particle | T.W | T.W |
| Edge | Soft edge (Airy) | Soft edge (Airy) |
| Global convergence | Semi circular | VKLS |

# Plan

1. Longest increasing subsequence and Ulam–Hammersley problem.
2. The first arrows of random Young tableaux (edge)
3. The Vershik-Kerov-Logan-Shepp shape

# Longest increasing subsequence and Ulam–Hammersley problem

**Conjecture (Ulam (1961))**

*If* $\sigma_n \sim U_{\mathfrak{S}_n}$, *then*

$$\lim_{n\to\infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} = c.$$

Proved by Hammersley.

# Longest increasing subsequence

**Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))**

*If* $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} = 2$$

*and*

$$\frac{\ell(\sigma_n)}{\sqrt{n}} \xrightarrow{\mathbb{P}} 2.$$

**Theorem (Baik, Deift, and Johansson (1999))**

*If* $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s).$$

| | GUE + Other random matrices | Uniform permutation |
|---|---|---|
| Largest particle | T.W | T.W |
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| Global convergence | Semi circular | VKLS |

# Longest increasing subsequence

## Theorem ( K (2018))

*Assume that the sequence of random permutations* $(\sigma_n)_{n\geq 1}$ *satisfies:*

- *For all positive integer* $n$*,* $\sigma_n$ *is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all* $\varepsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{H2}$$

*Then for all* $s \in \mathbb{R}$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s). \tag{TW}$$

# Ewens' case

**Definition (Ewens distribution)**

If $\sigma_n \sim Ew(\theta)$ then

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#(\sigma)}}{\prod_{k=0}^{n-1}(\theta + k)}.$$

- $\theta = 1$: uniform distribution.
- $\mathbb{E}(\#(\sigma_n)) = 1 + \sum_{k=1}^{n-1} \frac{\theta}{\theta+k} \sim \theta \log(n)$.

# Ewens' case

## Corollary

*Assume that* $\sigma_n \sim Ew(\theta_n)$. *If*

$$\lim_{n\to\infty} \frac{\theta_n \log(n)}{n^{\frac{1}{6}}} = 0. \tag{H'2}$$

*Then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s). \tag{TW}$$

Other applications: Ewens-Pitman, virtual permutations (Kingman), etc.

# Longest common subsequence

**Theorem ( K (2019))**

*Assume that*

- *For all positive integer $n$, $\sigma_n$ and $\rho_n$ are independent.*
- *For all positive integer $n$, $\sigma_n$ is invariant under conjugation i.e. $\forall \sigma, \rho \in \mathfrak{S}_n$,*

$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{1}$$

- *For all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{2}$$

*Then for all $s \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{LCS(\sigma_n, \rho_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \le s\right) = F_2(s). \tag{TW}$$

# Plan

1. Longest increasing subsequence and Ulam–Hammersley problem.
2. The first arrows of random Young tableaux (edge)
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# Young diagram

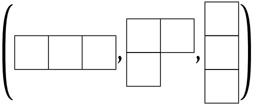
## Definition (Young diagram)

$\lambda = (\lambda_i)_{i \geq 1} \in \mathbb{N}^{\mathbb{N}^*}$ is a Young diagram of size $n$ if

- $\forall i \geq 1,\ \lambda_{i+1} \leq \lambda_i$,
- $\sum_{i=1}^{\infty} \lambda_i = n$.

Example: Young diagrams of size 3 are

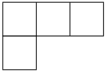
$\mathbb{Y}_3 = (3, \underline{0}), (2, 1, \underline{0}), (1, 1, 1, \underline{0})$

or .

# Young tableau

## Definition (Young tableau)

A Young tableau of shape $\lambda$ is a filling of the boxes of $\lambda$ using the entries $\{1, 2, \ldots, n\}$ and the entries in each row and each column are increasing.

- Example: Young tableaux of shape  are

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | | |

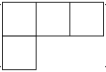
,

| | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | | |

,

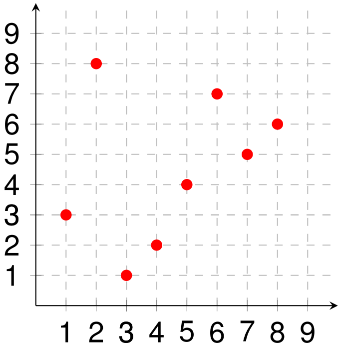
| | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | | |

.

- $dim(\lambda) = \#$ Young tableaux of shape $\lambda$.
- Example: $dim\Big($  $\Big) = 3$. $\mathfrak{S}_n$ indexed by $\lambda$.

# Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$



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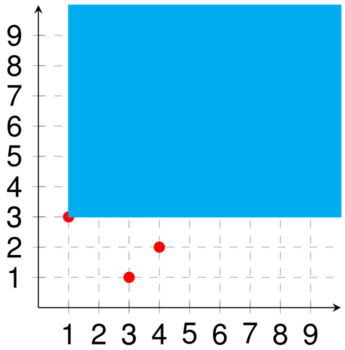

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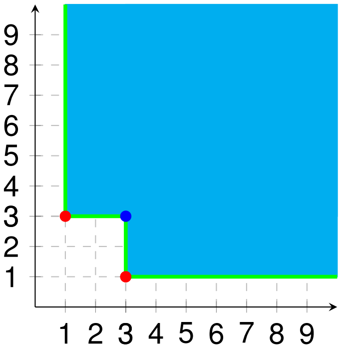

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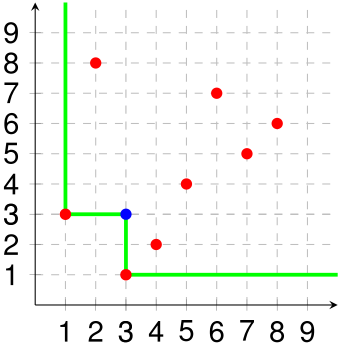

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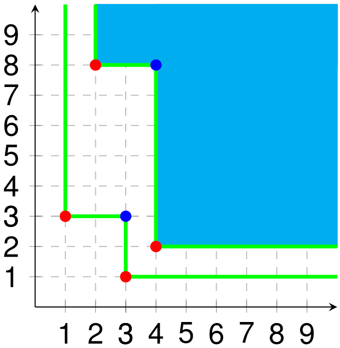

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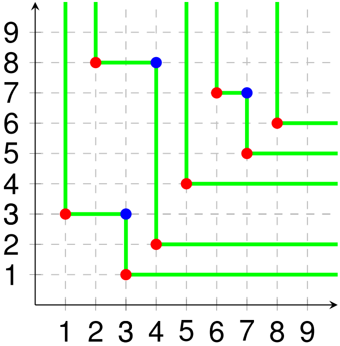
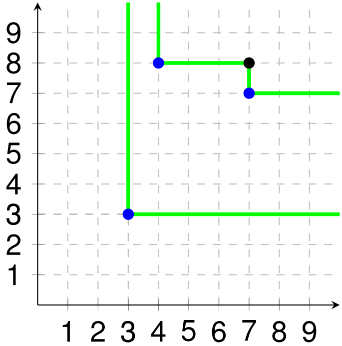
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 6 | , | 1 | 2 | 5 | 6 | 8 |

# Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$




| | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 6 |
| 3 | 7 | | | |

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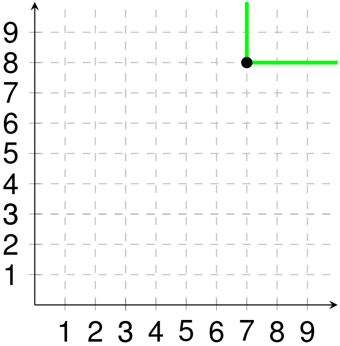
| | | | | |
|---|---|---|---|---|
| 1 | 2 | 5 | 6 | 8 |
| 3 | 4 | | | |

# Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$




| | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 6 |
| 3 | 7 | | | |
| 8 | | | | |

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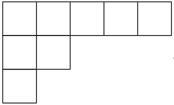
| | | | | |
|---|---|---|---|---|
| 1 | 2 | 5 | 6 | 8 |
| 3 | 4 | | | |
| 7 | | | | |

# Robinson-Schensted correspondence

- One-to-one correspondence between permutations and pairs of standard Young tableaux of the same shape.
- We denote by $\lambda(\sigma) := (\lambda_i(\sigma))_{i\geq 1}$ the shape of the image of $\sigma$ by this correspondence. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix} \quad \text{then} \quad \lambda(\sigma) =$$



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- $\ell(\sigma) = \lambda_1(\sigma)$.
- If $\sigma_n \sim U_{\mathfrak{S}_n}$ then $\lambda(\sigma_n) \sim PL_n$. For any $\mu \in \mathbb{Y}_n$,

$$\mathbb{P}(\lambda(\sigma_n) = \mu) = \frac{\#\{\text{pairs of Young tableaux of shape } \mu\}}{C} = \frac{dim(\mu)^2}{n!}.$$

# Plancherel measure

## Theorem

*Let* $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$ *be the eigenvalues of the GUE of size* $n$. $\forall k \geq 1$, $\forall s_1, s_2, \dots, s_k \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ (\lambda_i - \sqrt{2n})\sqrt{2}n^{\frac{1}{6}} \leq s_i\right) = \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i).$$

$\{\xi_1 \geq \xi_2 \geq \dots \geq \xi_k \geq \dots\}$: *Airy ensemble.*

## Theorem (Borodin, Okounkov, and Olshanski (2000))

*If* $\sigma_n \sim U_{\mathfrak{S}_n}$ *then* $\forall k \geq 1$, $\forall s_1, s_2, \dots, s_k \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i).$$

$\{\xi_1 \geq \xi_2 \geq \dots \geq \xi_k \geq \dots\}$: *Airy ensemble.*

# Edge: Plancherel case

| | GUE + Other random matrices | Uniform permutation |
|---|---|---|
| Largest particle | T.W | T.W |
| Edge | Soft edge (Airy) | Soft edge (Airy) |
| Global convergence | Semi circular | VKLS |

# Edge: generalization

## Theorem ( K (2018))

*Assume that the sequence of random permutations* $(\sigma_n)_{n\geq 1}$ *satisfies:*

- *For all positive integer* $n$*,* $\sigma_n$ *is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all* $\varepsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{H2}$$

*Then for all* $s \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i).$$

# Plan

1. Longest increasing subsequence and Ulam–Hammersley problem.
2. The first arrows of random Young tableaux (edge)
3. The Vershik-Kerov-Logan-Shepp shape

# Vershik-Kerov-Logan-Shepp shape

Consider the real-symmetric (or hermitian) $N \times N$ matrix $X_N$ with i.i.d entries and denote by $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_N$ its eigenvalues.

## Theorem (Wigner)

*Assume that:*

- $\mathbb{E}(x_{i,j}) = 0$
- $\mathbb{E}(x_{i,j}^2) = 1$
- $\forall k, \mathbb{E}(x_{i,j}^k) < \infty$

*then*

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{\lambda_i}{\sqrt{n}}} \xrightarrow{*} \mu_{sc},$$

*with*

$$d\mu_{sc}(u) := \frac{\sqrt{2-u^2}}{\pi} du.$$

# Russian notations

- Rotate the diagram by $\frac{3\pi}{4}$.
- Complete the high function by $x \to |x|$.
- We denote by $L_\lambda$ the resulting function.

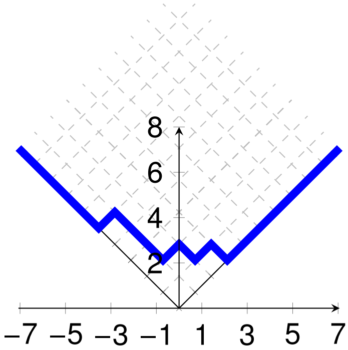



Figure: $L_{(5,2,1,\underline{0})}$

# Vershik-Kerov-Logan-Shepp shape

**Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))**

*If* $\sigma_n \sim U_{\mathfrak{S}_n}$*, then for any* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s) \right| < \varepsilon\right) = 1,$$

*where*

$$\Omega(s) := \begin{cases} \frac{2}{\pi}\left(s \arcsin(s) + \sqrt{1-s^2}\right) & \text{if } |s| < 1 \\ |s| & \text{if } |s| \geq 1 \end{cases}.$$

# Vershik-Kerov-Logan-Shepp shape

$\Omega$ is strongly related to the semi-circular law.
We denote by

$$\omega(s) := \frac{\Omega(2s) - |2s|}{2}.$$

We have

$$\exp\left(\int_{\mathbb{R}} \frac{d\omega(u)}{u - \frac{1}{x}}\right) = \int_{\mathbb{R}} \frac{d\mu_{sc}(u)}{1 - ux}.$$

# Vershik-Kerov-Logan-Shepp shape

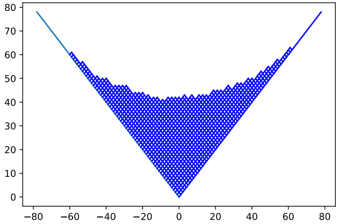



Figure: Typical Young diagram under the Plancherel distribution

| | GUE | Uniform permutation |
|---|---|---|
| Largest particle | T.W | T.W |
| Edge | Soft edge (Airy) | Soft edge (Airy) |
| Global convergence | Semi circular | VKLS |

# Limit shape

**Theorem ( K (2018))**

*Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ satisfies:*

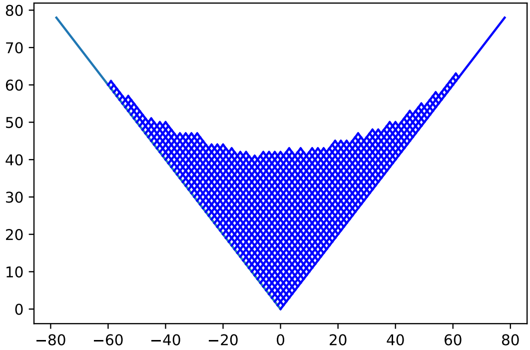
- *For all positive integer $n$, $\sigma_n$ is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all $\varepsilon > 0$,*
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n} > \varepsilon\right) = 0. \tag{H3}$$

*Then for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}}\left|\frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s)\right| < \varepsilon\right) = 1.$$

# Conclusion

| | GUE<br>+ Other RM) | Plancherel | Random permutations invariant under conjugation (with a good control on cycles' number) |
|---|---|---|---|
| Largest particle | T.W | T.W | T.W |
| Edge | Soft edge (Airy) | Soft edge (Airy) | Soft edge (Airy) |
| Global convergence | Semi circular | VKLS | VKLS |
| Fluctuations | Gaussian | Gaussian | ?? |
| Bulk | Sine process | Discrete sine process | ?? |




Thank you for your attention

# References


M. K. Monotonous subsequences and the descent process of invariant random permutations. *Electron. J. Probab.*, 23:31 pp., 2018. doi: 10.1214/18-EJP244. URL https://doi.org/10.1214/18-EJP244.

M. K. On the longest common subsequence of independent random permutations invariant under conjugation. *arXiv e-prints*, art. arXiv:1904.00725, Apr 2019.

J. Baik, P. Deift, and K. Johansson. On the Distribution of the Length of the Longest Increasing Subsequence of Random Permutations. *Journal of the American Mathematical Society*, 12(04):1119–1179, oct 1999.

A. Borodin and V. Gorin. Lectures on integrable probability. *arXiv e-prints*, art. arXiv:1212.3351, Dec 2012.

A. Borodin, A. Okounkov, and G. Olshanski. Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.*, 13 (3):481–515, 2000. ISSN 0894-0347. doi: 10.1090/S0894-0347-00-00337-4. URL https://doi.org/10.1090/S0894-0347-00-00337-4.

B. Logan and L. Shepp. A variational problem for random Young tableaux. *Advances in Mathematics*, 26(2):206 – 222, 1977. ISSN 0001-8708.

S. M. Ulam. Monte Carlo calculations in problems of mathematical physics. In *Modern mathematics for the engineer: Second series*, pages 261–281. McGraw-Hill, New York, 1961.

A. M. Vershik and S. V. Kerov. Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux. *Dokl. Akad. Nauk SSSR*, 233(6):1024–1027, 1977. ISSN 0002-3264.