Eulerian polynomials and excedance statistics

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Introduction

If $A = \{a_k\}_{k=0}^n$ is a finite sequence of real numbers, then

- A is unimodal if there is an index $0 \le j \le n$ such that

$$a_0 \le \cdots \le a_{j-1} \le a_j \ge a_{j+1} \ge \cdots \ge a_n.$$

- A is log-concave if

$$a_j^2 \ge a_{j-1}a_{j+1}, \quad \text{for all } 1 \le j \le n.$$

- the generating polynomial, $p_A(x) = a_0 + a_1x + \cdots + a_nx^n$, is called real rooted if all its zeros are real.
- the polynomial $p_A(x)$ or sequence A is called palindromic if $a_i = a_{n-i}$ for $0 \le j \le n/2$.

Example

The n-th row of Pascal's triangle $\{\binom{n}{k}\}_{k=0}^{n}$: $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{2}$, $\ldots$, $\binom{n}{n}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & \cdots & & \cdots & & \end{array}$$

The explicit formula $\binom{n}{k} = n!/k!(n-k)!$ implies

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.$$

A basic theorem

Let $A = \{a_k\}_{k=0}^n$ be a finite sequence of nonnegative numbers.

(a) If $p_A(x)$ is real-rooted, then the sequence $A' := \{a_k/\binom{n}{k}\}_{k=0}^n$ is log-concave.

(b) If A' is log-concave, then so is A.

(c) If A is log-concave and positive, then A is unimodal.

Remark

We say A is ultra-log-concave if A' is log-concave.

real-roots $\Longrightarrow$ *ultra-log-concavity* $\Longrightarrow$ *log-concavity* $\Longrightarrow$ *unimodality*

$$a_k^2 \geq a_{k-1}a_{k+1}\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right).$$

Unimodality and γ-positivity

The sequence $A = \{a_k\}_{k=0}^n$ is symmetric (or plindromic) with center of symmetry $n/2$ if $a_k = a_{n-k}$ for $0 \leq k \leq n/2$.
The linear space of polynomials $h(x) = \sum_{k=0}^n a_x x^k \in \mathfrak{R}[x]$ which are symmetric with center of symmetry $n/2$ has a basis

$$B_n := \{x^k(1+x)^{n-2k}\}_{k=0}^{\lfloor n/2 \rfloor}.$$

Indeed, we have the so-called Waring's formula:

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j}.$$

Proof of the symmetric expansion

Let $\gamma_{n,j} = (-1)^j \frac{n}{n-j}\binom{n-j}{j}$, then

$$x^n + y^n = \sum_{j=0}^{n/2} \gamma_{n,j}\,(xy)^j (x+y)^{n-2j}.$$

As $a_j = a_{n-j}$ (symmetry), we have

$$\begin{aligned} a_j x^j + a_{n-j} x^{n-j} &= a_j x^j (1 + x^{n-2j}) \\ &= a_j x^j \sum_k \gamma_{n-2j,k} x^k (x+1)^{n-2j-2k} \\ &= a_j \sum_k \gamma_{n-2j,k} x^{k+j} (x+1)^{n-2(j+k)} \end{aligned}$$

Real-roots and symmetry

If $h(x) = \sum_{k=0}^{n} \gamma_k x^k (1+x)^{n-2k}$ with $\gamma_k \geq 0$ we say that h is γ-nonnegative. Since the binomial coefficients are unimodal, having a nonnegative γ-vector implies unimodality of $\{a_k\}_{k=0}^{n}$.
Let Γ_+^n be the convex cone of polynomials that have nonnegative coefficients when expanded in B_n. Clearly

$$\Gamma_+^n \cdot \Gamma_+^m \subset \Gamma_+^{m+n}.$$

If $h(x) = \sum_{k=0}^{n} a_k x^k \in \mathfrak{R}_+[x]$ is symmetric with center of symmetry $n/2$. If all its zeros are real, then we may pair the negative zeros into reciprocal pairs

$$h(x) = Ax^k \prod_{l=1}^{\ell} (x+\theta_i)(x+1/\theta_i) = Ax^k \prod_{l=1}^{\ell} ((1+x)^2 + (\theta_i + 1/\theta_i - 2)x),$$

where $A > 0$. Since x and $(1+x)^2 + (\theta_i + 1/\theta_i - 2)x$ are in Γ_+^1, we see that $h(x)$ is γ-nonnegative.

Eulerian polynomials

The eulerian polynomials $A_n(t)$ can be defined by

$$\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}.$$

The first values of $A_n(t) := a_{n,1}t + a_{n,2}t^2 + \cdots + a_{n,n}t^n$:

$$A_1(t) = t,$$
$$A_2(t) = t + t^2,$$
$$A_3(t) = t + 4t^2 + t^3,$$
$$A_4(t) = t + 11t^2 + 11t^3 + t^4,$$
$$A_5(t) = t + 26t^2 + 66t^3 + 26t^4 + t^5.$$

Combinatorial interpretations

Let $\mathfrak{S}_n$ is the set of permutations on $\{1, \ldots, n\}$.

$$\begin{aligned}\operatorname{des}\sigma &= \#\{i \in [n-1] | \sigma(i) > \sigma(i+1)\},\\ \operatorname{exc}\sigma &= \#\{i \in [n] | \sigma(i) > i\},\\ \operatorname{wex}\sigma &= \#\{i \in [n] | \sigma(i) \geq i\}.\end{aligned}$$

Proposition

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{1+\operatorname{des}\sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{1+\operatorname{exc}\sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{wex}\sigma}.$$

Let $\tilde{A}(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{exc}\sigma}$. Then

$$\sum_{n=0}^{\infty} \tilde{A}(t) \frac{x^n}{n!} = \frac{1-t}{e^{(t-1)x} - t}.$$

An example for S_3

σ	des	exc	wex
123	0	0	3
132	1	1	2
213	1	1	2
231	1	2	2
312	1	1	1
321	2	1	2

Hence

$$A_3(t) = t + 4t^2 + t^3.$$

The equidistribution exc $\sim$ des can be explained by Foata's transformation. The mapping $\sigma \mapsto \tau$ defined by $\tau = \sigma(2)\ldots\sigma(n-1)\sigma(1)$ has the property that $\mathrm{wex}(\sigma) = 1 + \mathrm{exc}(\tau)$.

Unimodal and symmetric property

Proposition

The Eulerian polynomial $A_n(t)$ is symmetric and has only real roots.

Hence $A_n(t)$ is γ-positive:

$$\begin{aligned} t &= t, \\ t + t^2 &= t \cdot (1+t), \\ t + 4t^2 + t^3 &= t \cdot (1+t)^2 + 2t^2, \\ t + 11t^2 + 11t^3 + t^4 &= t(1+t)^3 + 8t^2(1+t). \end{aligned}$$

This formula implies both the symmetry and the unimodality of the Eulerian numbers.

Foata-Schützenberger, Foata-Strehl in 1970's

Let $\sigma = \sigma_1 \ldots \sigma_n \in \mathfrak{S}_n$ with $\sigma(0) = \sigma(n+1) = 0$. An integer $i \in [1, n]$ is

- a peak in σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- a double descent if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Proposition

Let $a_{n,k}$ be the number of permutations of $[n]$ with k peaks and without double descent. Then

$$A_n(t) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n+1-2k}. \tag{1}$$

Example: If $n = 4$, $k = 2$, then $a_{n,k} = 8$:

2134, 3124, 4123, 1423, 2413, 3412, 1324, 2314

peak polynomials

Let $\operatorname{pk}\pi$ be the number of peaks of π and define the peak polynomials

$$P_n^{\mathrm{pk}}(x) := \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{pk}\pi}.$$

Theorem (Stembridge 1997)

For $n \geq 1$, we have

$$A_n(t) = \left(\frac{1+t}{2}\right)^{n-1} P_n^{\mathrm{pk}}\left(\frac{4t}{(1+t)^2}\right).$$

It is easy to see that Stembridge's identity is equivalent to the γ-expansion formula of Eulerian polynomials.

peak-descent polynomials

Let

$$P_n^{(\mathrm{pk},\mathrm{des})}(x,t) := \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{pk}\,\pi} t^{\mathrm{des}\,\pi},$$

Theorem (Zhuang 2017)

For $n \geq 1$, we have

$$A_n(t) = \left(\frac{1+xt}{1+x}\right)^{n-1} P_n^{(\mathrm{pk},\mathrm{des})}\left(\frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt}\right).$$

Brandën's refinement of Eulerian polynomials

Let $\sigma = \sigma(1)\ldots\sigma(n) \in \mathfrak{S}_n$, define the statistics

$$(2\text{-}31)\sigma = \#\{(i,j) | 1 \leq i < j \leq n-1, \quad \sigma(j+1) < \sigma(i) < \sigma(j)\},$$
$$(13\text{-}2)\sigma = \#\{(i,j) | 2 \leq i < j \leq n, \quad \sigma(i-1) < \sigma(j) < \sigma(i)\}.$$

In 2008, Brändén (after the works of Postnikov, Williams and Corteel) considered the refined Eulerian polynomials:

$$A_n(p,q,t) := \sum_{\sigma \in \mathfrak{S}_n} p^{(13\text{-}2)\sigma} q^{(2\text{-}31)\sigma} t^{\mathrm{des}\sigma}$$

and proved, modifying the Foata-Strehl action, the identity:

$$A_n(p,q,t) := \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p,q) t^k (1+t)^{n-1-2k},$$

where $2^{n-1-2k} a_{n,k}(p,q) \in \mathbb{N}[p,q]$.

Our Theorem

Let

$$P_n^{(13\text{-}2,2\text{-}31,\mathrm{pk},\mathrm{des})}(p,q,x,t) := \sum_{\pi\in\mathfrak{S}_n} p^{(13\text{-}2)\,\pi} q^{(2\text{-}31)\,\pi} x^{\mathrm{pk}\,\pi} t^{\mathrm{des}\,\pi}.$$

Theorem

For $n \geq 1$, we have

$$A_n(p,q,t) = \left(\frac{1+xt}{1+x}\right)^{n-1} P_n^{(13\text{-}2,2\text{-}31,\mathrm{pk},\mathrm{des})}\left(p,q,\frac{(1+x)^2 t}{(x+t)(1+xt)},\frac{x+t}{1+xt}\right).$$

Definitions

Definition

For $\sigma \in \mathfrak{S}_n$, a value $x = \sigma(i)$ $(i \in [n])$ is called

- *a cyclic peak if $i = \sigma^{-1}(x) < x$ and $x > \sigma(x)$;*
- *a cyclic valley if $i = \sigma^{-1}(x) > x$ and $x < \sigma(x)$;*
- *a double excedance if $i = \sigma^{-1}(x) < x$ and $x < \sigma(x)$;*
- *a double drop if $i = \sigma^{-1}(x) > x$ and $x > \sigma(x)$;*
- *a fixed point if $x = \sigma(x)$.*

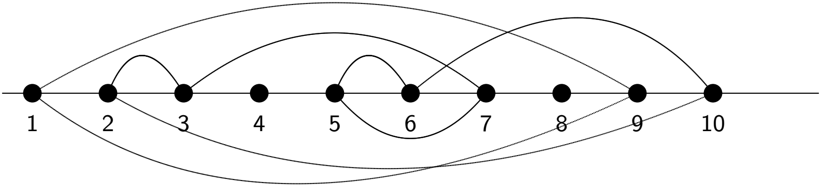
Let $\operatorname{cpk}\sigma$ (resp. $\operatorname{cvalley}\sigma$, $\operatorname{cda}\sigma$, $\operatorname{cdd}\sigma$, $\operatorname{fix}\sigma$) be the number of cyclic peaks (resp. valleys, double excedances, double drops, fixed points) in σ.

For $\sigma \in \mathfrak{S}_n$ define the crossing and nesting numbers by

$$\text{cros}(\sigma) = \sum_{i=1}^{n} \#\{j \in [n] | j < i \leq \sigma(j) < \sigma(i) \text{ or } j > i > \sigma(j) > \sigma(i)\},$$

$$\text{nest}(\sigma) = \sum_{i=1}^{n} \#\{j \in [n] | i < j \leq \sigma(j) < \sigma(i) \text{ or } i > j > \sigma(j) > \sigma(i)\}.$$

Diagram of $\sigma = 9\ 3\ 7\ 4\ 6\ 10\ 5\ 8\ 1\ 2$: $\text{cros}\,\sigma = 5$ and $\text{nest}\,\sigma = 10$.



Derangement analogue

Let $\mathfrak{D}_n$ be the set of *derangements* in $\mathfrak{S}_n$. Consider the derangement analogue of the Eulerian polynomials:

$$D_n(t) = \sum_{\sigma \in \mathfrak{D}_n} t^{\operatorname{exc}\sigma} = D_{n,1}t + D_{n,2}t^2 + \cdots + D_{n,n-1}t^{n-1}.$$

Proposition (Shin-Zeng 2012)

Let $b_{n,k}$ be the number of derangements of $[n]$ with k cyclic peaks and without cyclic double descents. Then

$$D_n(t) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} b_{n,k} t^k (1+t)^{n-2k}. \tag{2}$$

Derangement polynomials

Define the cpeak polynomials

$$P_n^{\mathrm{cpk}}(t) := \sum_{\pi \in \mathfrak{D}_n} t^{\mathrm{cpk}\,\pi}.$$

Then Shin-Zeng's formula reads

$$D_n(t) = \left(\frac{1+t}{2}\right)^n P_n^{\mathrm{cpk}}\left(\frac{4t}{(1+t)^2}\right).$$

Our Theorem

Define the following polynomial

$$P_n^{(\mathrm{cpk},\mathrm{exc})}(x,t) := \sum_{\pi\in\mathfrak{D}_n} x^{\mathrm{cpk}\,\pi} t^{\mathrm{exc}\,\pi}.$$

Theorem

For $n \geq 1$, we have

$$D_n(t) = \left(\frac{1+xt}{1+x}\right)^n P_n^{(\mathrm{cpk},\mathrm{exc})}\left(\frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt}\right).$$

Our Theorem

Let

$$D_n(q,t) = \sum_{\pi \in \mathfrak{D}_n} q^{\mathrm{nest}\,\pi} t^{\mathrm{exc}\,\pi},$$

$$P_n^{(\mathrm{nest},\mathrm{cpk},\mathrm{exc})}(x,t) = \sum_{\pi \in \mathfrak{D}_n} q^{\mathrm{nest}\,\pi} x^{\mathrm{cpk}\,\pi} t^{\mathrm{exc}\,\pi}.$$

Theorem

For all positive integers n and for each statistic stat $\in \{\mathrm{nest}\}$

$$D_n(q,t) = \left(\frac{1+xt}{1+x}\right)^n P_n^{(\mathrm{nest},\mathrm{cpk},\mathrm{exc})}\left(q, \frac{(1+x)^2 t}{(x+t)(1+xt)}, \frac{x+t}{1+xt}\right).$$

Two continued fractions of Euler, Rogers

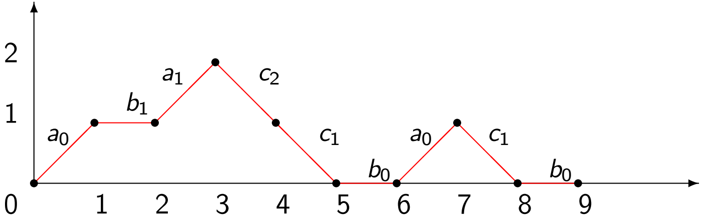
$$\sum_{n=0}^{\infty} (n+1)! x^n = \cfrac{1}{1-2x-\cfrac{1\cdot 2x^2}{1-4x-\cfrac{2\cdot 3x^2}{\cdots}}},$$

$$\sum_{n=0}^{\infty} n! x^n = \cfrac{1}{1-x-\cfrac{1^2x^2}{1-3x-\cfrac{2^2x^2}{\cdots}}}.$$

These formulas are related to the moment sequence of the Laguerre polynomials.

Combinatorial version by Flajolet, Françon-Viennot

Consider the following Motzkin path γ :



The weight is $w(\gamma) = a_0^2\, a_1 b_0^2\, b_1\, c_1^2\, c_2$. Denote by $\mathfrak{M}_n$ the set of Motzkin paths of length $n \geq 1$. Then

$$1 + \sum_{n\geq 1} \sum_{\gamma \in \mathfrak{M}_n} w(\gamma) x^n = \frac{1}{1 - b_0 x - \dfrac{a_0 c_1 x^2}{1 - b_1 x - \dfrac{a_1 c_2 x^2}{\cdots}}}. \qquad (3)$$

Laguerre history: $h = (\gamma; p_1, \ldots, p_n)$ with respect to the valuation

$$a_k = k+1, \quad b_k = 2k+2 \quad \text{for} \quad k \geq 0; \quad c_k = k+1 \quad \text{for} \quad k \geq 1.$$

The number of Laguerre histories of length n is $(n+1)!$.
Restricted Laguerre history: $h = (\gamma; p_1, \ldots, p_n)$ with respect to the valuation

$$a_k = k+1, \quad b_k = 2k+1 \quad \text{for} \quad k \geq 0; \quad c_k = k \quad \text{for} \quad k \geq 1.$$

The number of Restricted Laguerre histories of length n is $n!$.

Two theorems due to Françon-Viennot, Foata-Zeilberger, Biane

Theorem

There are bijections between $\mathfrak{S}_{n+1}$ and the Laguerre histories of length n.

Theorem

There are Bijections between $\mathfrak{S}_n$ and the Restricted Laguerre histories of length n.

Modified Foata-Strehl action

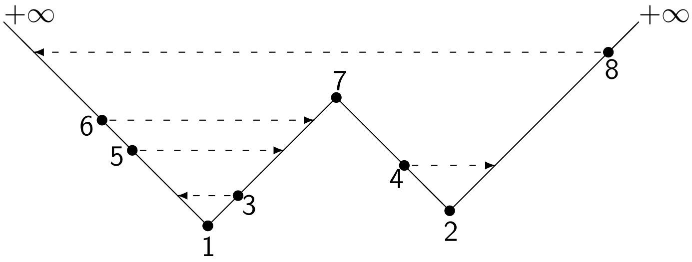


Figure: MFS-actions on 65137428

Let $\hat{\pi}$ be permutation with fixed Peaks and Valleys. Then

$$\sum_{\pi \in Orb(\hat{\pi})} t^{\mathrm{des}(\pi)} = t^{\mathrm{pk}(\hat{\pi})}(1+t)^{n-1-2\mathrm{pk}(\hat{\pi})}.$$

Merci de votre attention