Infinite energy solutions to the Navier-Stokes equations

Pedro Fernandez

Laboratoire de Mathématiques et Modélisation d'Evry

Interactions

May, 2019

1. The beggining
2. Self-similar solutions
3. Discretely self-similar solutions
4. How to use the energy controls ?

Uniformly locally square integrable solutions

Infinite energy solutions to the Navier-Stokes equations were introduced by Lemarié-Rieusset in 1999. This has allowed to show the existence of local weak solutions for a uniformly locally square integrable initial data.

Other constructions of infinite-energy solutions for locally uniformly square integrable initial data were given in 2006 by Basson [1] and in 2007 by Kikuchi and Seregin [6].

Uniformly locally square integrable solutions

Theorem

Let $\mathbf{u}_0 \in L^2_{uloc}$ *with* $\nabla \cdot \mathbf{u}_0 = 0$ *and* $\mathbb{F} \in (L^2_t L^2_x)_{uloc}((0,1) \times \mathbb{R}^3)$*. Then there exist a solution* $\mathbf{u}$ *to*

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

on $(0,T) \times \mathbb{R}^3$ *with*

$$T = \min(c, c(\|u_0\|_{L^2_{uloc}} + \|\mathbb{F}\|_{(L^2L^2)_{uloc}})^{-4})$$

and $\mathbf{u} \in (L^\infty_t L^2_x)_{uloc} \cap (L^2_t H^1_x)_{uloc}$ *satisfies*

Uniformly locally square integrable solutions

$$\sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^3}\Big(\int_{|x-y|<1}|\mathbf{u}(t,y)|^2\,dy\Big)^{\frac{1}{2}}\leq 2(\|\mathbf{u}_0\|_{L^2_{uloc}}+c\|\mathbb{F}\|_{(L^2L^2)_{uloc}})$$

$$\sup_{x\in\mathbb{R}^3}\Big(\int_0^T\int_{|x-y|<1}|\nabla\mathbf{u}(t,y)|^2\,dy\,ds\Big)^{\frac{1}{2}}\leq 2(\|\mathbf{u}_0\|_{L^2_{uloc}}+c\|\mathbb{F}\|_{(L^2L^2)_{uloc}})$$

Moreover, $\mathbf{u}$ *is suitable :* $\mathbf{u}$ *satisfies in* $\mathcal{D}'$ *the energy inequality*

$$\partial_t\Big(\frac{|\mathbf{u}|^2}{2}\Big)\leq\Delta\Big(\frac{|\mathbf{u}|^2}{2}\Big)-|\nabla\mathbf{u}|^2-\nabla\cdot\left(\Big(\frac{|\mathbf{u}|^2}{2}+p\Big)\mathbf{u}\right)+\mathbf{u}\cdot(\nabla\cdot\mathbb{F})$$

with $\nabla p=(Id-\mathbb{P})\nabla\cdot(\mathbb{F}-\mathbf{u}\otimes\mathbf{u})$.

For every compact subset K, $\lim_{t\to 0^+}\int_K|\mathbf{u}(t,x)-\mathbf{u}_0(x)|^2\,dx=0$

Self-similar solutions

The infinite energy solutions allowed Jia and Sverak [5] to construct in 2014 the self-similar solutions for large smooth data.

Definition

Let $\mathbf{u}_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. We say that $\mathbf{u}_0$ is a self-similar function (SS) if for all $\lambda > 1$, $\lambda \mathbf{u}_0(\lambda x) = \mathbf{u}_0$.
A vector field $\mathbf{u} \in L^2_{\mathrm{loc}}([0, +\infty) \times \mathbb{R}^3)$ is SS if for all $\lambda > 1$, $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t, x)$.
A forcing tensor $\mathbb{F} \in L^2_{\mathrm{loc}}([0, +\infty) \times \mathbb{R}^3)$ is SS if for all $\lambda > 1$, $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t, x)$.

Self-similar solutions

Their result has been extended in 2016 by Lemarié-Rieusset [9] to solutions for rough locally square integrable data. We remark that an homogeneous (of degree -1) and locally square integrable data is automatically uniformly locally L^2.

If $\mathbf{u}_0$ *is self-similar, we have* $\mathbf{u}_0(x) = \frac{1}{|x|}\mathbf{u}_0(\frac{x}{|x|})$. *From this equality, we find that, for* $\lambda > 1$

$$\int_{1<|x|<\lambda} |\mathbf{u}_0(x)|^2 \, dx = (\lambda - 1) \int_{S^2} |\mathbf{u}_0(\sigma)|^2 \, d\sigma$$

Discretely self-similar solutions

In 2018, Bradshaw and Tsai considered the case of self-similar solutions to a discrete subgroup of dilations. A locally L^2 initial data is not necessarily uniformly locally L^2, therefore their results are not consequence of constructions described by Lemarié-Rieusset in [8]. What can we do ?

Definition

Let $\mathbf{u}_0 \in L^2_{\text{loc}}(\mathbb{R}^3)$. We say that $\mathbf{u}_0$ is a λ-discretely self-similar function (λ-DSS) if there exists $\lambda > 1$ such that $\lambda \mathbf{u}_0(\lambda x) = \mathbf{u}_0$.
A vector field $\mathbf{u} \in L^2_{\text{loc}}([0, +\infty) \times \mathbb{R}^3)$ is λ-DSS if there exists $\lambda > 1$ such that $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t, x)$.
A forcing tensor $\mathbb{F} \in L^2_{\text{loc}}([0, +\infty) \times \mathbb{R}^3)$ is λ-DSS if there exists $\lambda > 1$ such that $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t, x)$.

I work with Pierre-Gilles Lémarié-Rieusset to give a new proof of the results of Chae and Wolf [3] and Bradshaw and Tsai [2] on the existence of λ-DSS solutions of the Navier–Stokes problem (and of Jia and Šverák [5] for self-similar solutions).

The procedure to obtain infinite energy solutions

For uniformly locally L^2 initial data, the idea is to solve the approximated Navier-Stokes problem using a point fix theorem, and then to prove that this approximated solutions satisfy a priory controls uniformly.

There exist $C > 0$ *such that for all* $0 < \varepsilon < 1$

$$\|\mathbf{u}_\varepsilon(t,.)\|^2_{L^2_{uloc}} + \frac{1}{2} sup_{x\in\mathbb{R}^3} \int_0^t \int_{B(x,1)} |\nabla \mathbf{u}_\varepsilon|^2 \, ds$$
$$\leq \|\mathbf{u}_0\|^2_{L^2_{uloc}} + C \int_0^t \|\mathbf{u}_\varepsilon(t,.)\|^2_{L^2_{uloc}} + \|\mathbf{u}_\varepsilon(t,.)\|^6_{L^2_{uloc}} \, ds$$

and the initial estimate of existence time permit to iterate the point fix while the quantity in left side is finite.

The procedure to obtain infinite energy solutions

A Grönwall non linear lemma permit to obtain a uniform existence time and we have the souhaitable controls to use Rellich-Lions Lemma to obtain a suite $\mathbf{u}_{\varepsilon_n}$ *which converges to a solution.*

In the case of discretely self-similar initial data $\mathbf{u}_0$*, we have that* $\mathbf{u}_0 \in L^2((1+|x|)^{-1-\varepsilon}dx)$*, where* $\varepsilon > 0$*. In this case, if we search solutions* $\mathbf{u}$ *in* $L^\infty_t L^2((1+|x|)^{-1-\varepsilon}dx)$ *and* $\nabla \mathbf{u} \in L^2_t L^2((1+|x|)^{-1-\varepsilon}dx)$ *we can not use directly a Piccard point fix theorem because the bi-linear form*

$$\int_0^t h_{t-s} * \nabla \cdot (\mathbf{u} \otimes (v * \theta_\varepsilon))(x,s)ds$$

is not continuous.

The procedure to obtain discretely self-similar solutions

To obtain approximated discretely self-similar solutions we need to use a mollifiers $\theta_{\sqrt{t}\varepsilon}$ to regularize the problem, and even if we use initial data in L^2 the bi-linear form with this mollifier is not continuous.

We obtain the a priori estimates and we construct solutions of the linearized problem, then we obtain solutions of the approximated problem by using the Leray-Schauder theorem

We prove the discretely self-similar property by demonstrate a uniqueness result for the approximated problem

The Rellich–Lions theorem permit to obtain a solution by taking the limit of a sequence of approximated solutions.

Thank you for your attention !

Bibliographie

A. Basson, *Solutions spatialement homogènes adaptées des équations de Navier–Stokes*, Thèse, Université d'Évry, 2006.

Z. Bradshaw and T.P. Tsai, *Discretely self-similar solutions to the Navier-Stokes equations with data in* L^2_{loc}, to appear in Analysis and PDE.

D. Chae and J. Wolf, *Existence of discretely self-similar solutions to the Navier-Stokes equations for initial value in* $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), 1019–1039..

L. Grafakos, *Classical harmonic analysis (2nd ed.)*, Springer, 2008.

H. Jia and V. Šverák, *Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions*, Invent. Math. 196 2014, 233–265.

N. Kikuchi and G. Seregin, *Weak solutions to the Cauchy problem for the Navier–Stokes equations satisfying the local energy inequality*, in Nonlinear equations and spectral theory. Amer. Math. Soc. Transl. Ser. 2, 220, M.S. Birman and N.N. Uraltseva eds., 2007, 141–164.

P.G. Lemarié–Rieusset, *Solutions faibles d'énergie infinie pour les équations de Navier–Stokes dans* $\mathbb{R}^3$, C. R. Acad. Sci. Paris, Serie I. 328 (1999), 1133–1138.

P.G. Lemarié-Rieusset, *Recent developments in the Navier–Stokes problem*, CRC Press, 2002.

P.G. Lemarié–Rieusset, *The Navier–Stokes problem in the 21st century*, Chapman & Hall/CRC, (2016).

J. Leray, *Essai sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math. 63 (1934), 193–248.