Raking-Ratio empirical process

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Introduction

Notation

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If $\mathcal{F}$ is a Donsker class, $\alpha_n \xrightarrow[n\to+\infty]{\mathcal{L}} \mathbb{G}$ in $l^\infty(\mathcal{F})$ where $\mathbb{G}$ is the P-brownian bridge, i.e the Gaussian process with covariance

$$\mathrm{Cov}(\mathbb{G}(f), \mathbb{G}(g)) = P(fg) - P(f)P(g).$$

Plan

1. Raking-ratio method

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2. Extension 1: re-sampling method with auxiliary information

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2. Extension 1: re-sampling method with auxiliary information
3. ~~Extension 2: auxiliary information learning~~

Raking-ratio method

Example of the Raking-Ratio method

Literature: Deming/Stephan, Sinkhorn, Ireland/Kullback.

Description:

	$A_1^{(2)}$	$A_2^{(2)}$	$A_3^{(2)}$	$\mathbb{P}_n[\mathcal{A}^{(1)}]$	$P[\mathcal{A}^{(1)}]$
$A_1^{(1)}$	0.2	0.25	0.1	0.55	0.52
$A_2^{(1)}$	0.1	0.2	0.15	0.45	0.48
$\mathbb{P}_n[\mathcal{A}^{(2)}]$	0.3	0.45	0.25	1	
$P[\mathcal{A}^{(1)}]$	0.31	0.4	0.29		

We have a table of frequencies whose margins do not correspond to known margins. The algorithm proposes to correct this

Example of the Raking-Ratio method

	$A_1^{(2)}$	$A_2^{(2)}$	$A_3^{(2)}$	$\mathbb{P}_n^{(1)}[\mathcal{A}^{(1)}]$	$P[\mathcal{A}^{(1)}]$
$A_1^{(1)}$	0.189	0.236	0.095	0.52	0.52
$A_2^{(1)}$	0.11	0.21	0.16	0.48	0.48
$\mathbb{P}_n^{(1)}[\mathcal{A}^{(2)}]$	0.299	0.446	0.255	1	
$P[\mathcal{A}^{(2)}]$	0.31	0.4	0.29		

The totals for each line are first corrected by applying a rule of three. Each cell is multiplied by the ratio of the expected total of each line on the total of each line.

Example of the Raking-Ratio method

	$A_1^{(2)}$	$A_2^{(2)}$	$A_3^{(2)}$	$\mathbb{P}_n^{(2)}[\mathcal{A}^{(1)}]$	$P[\mathcal{A}^{(1)}]$
$A_1^{(1)}$	0.196	0.212	0.108	0.516	0.52
$A_2^{(1)}$	0.114	0.188	0.182	0.484	0.48
$\mathbb{P}_n^{(2)}[\mathcal{A}^{(2)}]$	0.31	0.4	0.29	1	
$P[\mathcal{A}^{(2)}]$	0.31	0.4	0.29		

The same reasoning is applied to correct the totals for each column.
These last two operations are repeated in a loop.

Example of the Raking-Ratio method

	$A_1^{(2)}$	$A_2^{(2)}$	$A_3^{(2)}$	$\mathbb{P}_n^{(\infty)}[\mathcal{A}^{(1)}]$	$P[\mathcal{A}^{(1)}]$
$A_1^{(1)}$	0.199	0.212	0.109	0.52	0.52
$A_2^{(1)}$	0.111	0.188	0.181	0.48	0.48
$\mathbb{P}_n^{(\infty)}[\mathcal{A}^{(2)}]$	0.31	0.4	0.29	1	
$P[\mathcal{A}^{(2)}]$	0.31	0.4	0.29		

Very quickly the algorithm stabilizes. Totals are the expected totals. For this example it took only 7 iterations.

Remark: we can rake on more than two partitions!

Notation of Raking-Ratio method

In turn N the algorithm does:

$$p^{(N+1)}(A) = \sum_{j=1}^{m_{N+1}} p^{(N)}(A \cap A_j^{(N+1)}) \frac{P(A_j^{(N+1)})}{p^{(N)}(A_j^{(N+1)})}.$$

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We define the raked empirical measure $\mathbb{P}_n^{(N)}$ to be $\mathbb{P}_n^{(0)} = \mathbb{P}_n$ and

$$\mathbb{P}_n^{(N+1)}(f) = \sum_{j=1}^{m_{N+1}} \mathbb{P}_n^{(N)}(f \mathbf{1}_{A_j^{(N+1)}}) \frac{P(A_j^{(N+1)})}{\mathbb{P}_n^{(N)}(A_j^{(N+1)})}.$$

In particular, $\mathbb{P}_n^{(N+1)}(A_j^{(N+1)}) = P(A_j^{(N+1)}), \forall j = 1, \ldots, m_{N+1}$.

Notation of Raking-Ratio method

Let $\alpha_n^{(N)}(f) = \sqrt{n}(\mathbb{P}_n^{(N)}(f) - P(f))$ the raked empirical process.

$$\alpha_n^{(N+1)}(f) = \sum_{j \leqslant m_{N+1}} \frac{P(A_j^{(N+1)})}{\mathbb{P}_n^{(N)}(A_j^{(N+1)})} \left(\alpha_n^{(N)}(f \mathbf{1}_{A_j^{(N+1)}}) - \mathbb{E}[f|A_j^{(N+1)}] \alpha_n^{(N)}(A_j^{(N+1)}) \right)$$

with $\mathbb{E}[f|A] = \frac{P(f\mathbf{1}_A)}{P(A)}$.

In particular, $\alpha_n^{(N+1)}(A_j^{(N+1)}) = 0, \quad \forall j = 1, \ldots, m_{N+1}$.

Remark: $\mathbb{E}[\alpha_n^{(N)}(f)] \neq 0 \Rightarrow \alpha_n^{(N)}$ is no more centered.

Raking-Ratio method

Goals

- Weak convergence in $\ell^\infty(\mathcal{F})$ of $\alpha_n^{(N)}(\mathcal{F})$ when $n \to +\infty$ towards a centered Gaussian process $\mathbb{G}^{(N)}(\mathcal{F})$;

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- Variance of $\mathbb{G}^{(N)}(f)$: is it lower than that of $\mathbb{G}$? If a loop is performed with the Raking-Ratio method, does the variance decrease with each loop turn?

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Goals

- Weak convergence in $\ell^\infty(\mathcal{F})$ of $\alpha_n^{(N)}(\mathcal{F})$ when $n \to +\infty$ towards a centered Gaussian process $\mathbb{G}^{(N)}(\mathcal{F})$;
- Variance of $\mathbb{G}^{(N)}(f)$: is it lower than that of $\mathbb{G}$? If a loop is performed with the Raking-Ratio method, does the variance decrease with each loop turn?
- If we rake only two partitions, what's the limit of $\alpha_n^{(N)}(\mathcal{F})$ as $n, N \to +\infty$?

Main result

Weak convergence of $\alpha_n^{(N)}$

Under some entropy conditions on $\mathcal{F}$,

$$(\alpha_n^{(0)}, \dots, \alpha_n^{(N_0)}) \xrightarrow[n \to +\infty]{\mathcal{L}} (\mathbb{G}^{(0)}, \dots, \mathbb{G}^{(N_0)}) \quad \text{in} \quad \ell^\infty(\mathcal{F}^{N_0} \to \mathbb{R}^{N_0})$$

with $\mathbb{G}^{(N)}$ the Gaussian process defined by

$$\mathbb{G}^{(0)} = \mathbb{G} \quad \text{and} \quad \mathbb{G}^{(N+1)}(f) = \mathbb{G}^{(N)}(f) - \sum_{j=1}^{m_{N+1}} \mathbb{E}[f|A_j^{(N+1)}]\mathbb{G}^{(N)}(A_j^{(N+1)})$$

Weak convergence of $\alpha_n^{(N)}$

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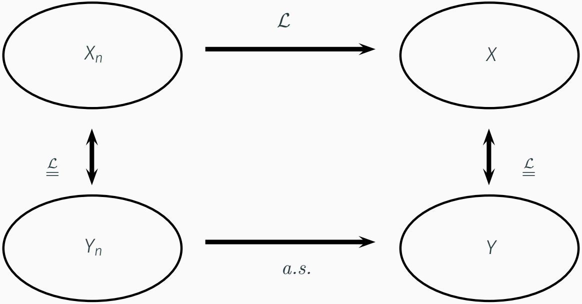
Recall that

$$\alpha_n^{(N+1)}(f) = \sum_{j\leqslant m_{N+1}} \frac{P(A_j^{(N+1)})}{\mathbb{P}_n^{(N)}(A_j^{(N+1)})} \left(\alpha_n^{(N)}(f\mathbf{1}_{A_j^{(N+1)}}) - \mathbb{E}[f|A_j^{(N+1)}]\alpha_n^{(N)}(A_j^{(N+1)})\right)$$

Main tool

Idea of strong approximation



Results: KMT, Berthet-Mason.

Strong approximation of $\alpha_n^{(N)}(\mathcal{F})$

Under some entropy conditions on $\mathcal{F}$ we can construct on the same probability space $X_1, \ldots, X_n$ and a version $\mathbb{G}_n^{(N)}$ of $\mathbb{G}^{(N)}$ such that for large n,

$$\mathbb{P}\left(\max_{0 \leqslant N \leqslant N_0} ||\alpha_n^{(N)} - \mathbb{G}_n^{(N)}||_{\mathcal{F}} > Cv_n\right) \leqslant \frac{1}{n^2},$$

with $v_n \to 0$.

By Borell-Cantelli,

$$\max_{0 \leqslant N \leqslant N_0} ||\alpha_n^{(N)} - \mathbb{G}_n^{(N)}||_{\mathcal{F}} = O_{\text{a.s.}}(v_n).$$

Consequences of strong approximation

Uniform estimation of bias and variance of Raking-Ratio method

Under some entropy conditions on $\mathcal{F}$, there exists $C > 0$ such that

$$\limsup_{n\to+\infty} \frac{\sqrt{n}}{v_n} \max_{0\leqslant N\leqslant N_0} \sup_{f\in\mathcal{F}} \left| \mathbb{E}[\mathbb{P}_n^{(N)}(f)] - P(f) \right| \leqslant C,$$

$$\limsup_{n\to+\infty} \frac{n}{v_n} \sup_{f\in\mathcal{F}} \left| \mathrm{Var}(\mathbb{P}_n^{(N)}(f)) - \frac{1}{n}\mathrm{Var}(\mathbb{G}^{(N)}(f)) \right| \leqslant C.$$

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$$\limsup_{n\to+\infty} \frac{n}{v_n} \sup_{f\in\mathcal{F}} \left|\mathrm{Var}(\mathbb{P}_n^{(N)}(f)) - \frac{1}{n}\mathrm{Var}(\mathbb{G}^{(N)}(f))\right| \leqslant C.$$

Uniform Berry-Esseen bound

Under some entropy conditions on $\mathcal{F}$,

$$\max_{0\leqslant N\leqslant N_0} \sup_{f\in\mathcal{F}} \sup_{x\in\mathbb{R}} \left|\mathbb{P}(\alpha_n^{(N)}(f) \leqslant x) - \mathbb{P}(\mathbb{G}^{(N)}(f) \leqslant x)\right| \leqslant C v_n.$$

Raking-Ratio results

We denote

- $\mathbb{E}[f|\mathcal{A}^{(k)}] = (\mathbb{E}[f|A_1^{(k)}], \ldots, \mathbb{E}[f|A_{m_k}^{(k)}])^t$;
- $\mathbb{G}[\mathcal{A}^{(k)}] = (\mathbb{G}(A_1^{(k)}), \ldots, \mathbb{G}(A_{m_k}^{(k)}))^t$;
- $(\mathsf{P}_{\mathcal{A}^{(k)}|\mathcal{A}^{(l)}})_{i,j} = P(A_j^{(k)}|A_i^{(l)})$.

Expression of $\mathbb{G}^{(N)}$

For all $N \in \mathbb{N}^*$ and $f \in \mathcal{F}$ it holds

$$\mathbb{G}^{(N)}(f) = \mathbb{G}(f) - \sum_{k=1}^{N} \Phi_k^{(N)}(f)^t \cdot \mathbb{G}[\mathcal{A}^{(k)}]$$

where

$$\Phi_k^{(N)}(f) = \mathbb{E}[f|\mathcal{A}^{(k)}] + \sum_{\substack{1 \leqslant L \leqslant N-k \\ k < l_1 < \cdots < l_L \leqslant N}} (-1)^L \mathsf{P}_{\mathcal{A}^{(l_1)}|\mathcal{A}^{(k)}} \mathsf{P}_{\mathcal{A}^{(l_2)}|\mathcal{A}^{(l_1)}} \ldots \mathsf{P}_{\mathcal{A}^{(l_L)}|\mathcal{A}^{(l_L-1)}} \cdot \mathbb{E}[f|\mathcal{A}^{(l_L)}].$$

Raking-Ratio results

We denote $(\mathrm{Var}((X_1,\ldots,X_n)^t))_{i,j} = \mathrm{Cov}(X_i,X_j)$

Variance and covariance of $\mathbb{G}^{(N)}$

For all $N \in \mathbb{N}^*$ and $f,g \in \mathcal{F}$ it holds

$$\mathrm{Var}(\mathbb{G}^{(N)}(f)) = \mathrm{Var}(\mathbb{G}(f)) - \sum_{k=1}^{N} \Phi_k^{(N)}(f)^t \cdot \mathrm{Var}(\mathbb{G}[\mathcal{A}^{(k)}]) \cdot \Phi_k^{(N)}(f)$$

$$\begin{aligned}\mathrm{Cov}(\mathbb{G}^{(N)}(f), \mathbb{G}^{(N)}(g)) &= \mathrm{Cov}(\mathbb{G}(f), \mathbb{G}(g)) \\ &\quad - \sum_{k=1}^{N} \mathrm{Cov}\left(\Phi_k^{(N)}(f)^t \cdot \mathbb{G}[\mathcal{A}^{(k)}], \Phi_k^{(N)}(g)^t \cdot \mathbb{G}[\mathcal{A}^{(k)}]\right)\end{aligned}$$

Raking-Ratio results

Corollary 1

For any $N \in \mathbb{N}$ and $f \in \mathcal{F}$, $\mathrm{Var}(\mathbb{G}^{(N)}(f)) \leqslant \mathrm{Var}(\mathbb{G}(f))$.

For any $\{f_1, \ldots, f_m\} \in \mathcal{F}$, $\Sigma_m - \Sigma_m^{(N)}$ is positive definite with

$$\Sigma_n^{(N)} = \mathrm{Var}((\mathbb{G}^{(N)}(f_1), \ldots, \mathbb{G}^{(N)}(f_m))^t),$$
$$\Sigma_n = \mathrm{Var}((\mathbb{G}(f_1), \ldots, \mathbb{G}(f_m))^t).$$

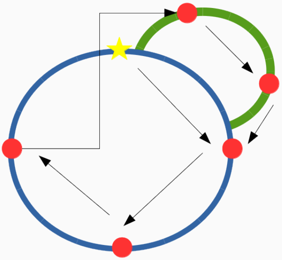
Raking-Ratio results

Corollary 2

Let $N_0, N_1 \in \mathbb{N}$ s.t. $N_1 \geqslant 2N_0$ and

$$\mathcal{A}^{(N_0-i)} = \mathcal{A}^{(N_1-i)}, \quad \forall 0 \leqslant i < N_0.$$

Then for all $f \in \mathcal{F}$, $\mathrm{Var}(\mathbb{G}^{(N_1)}(f)) \leqslant \mathrm{Var}(\mathbb{G}^{(N_0)}(f))$.



We work with $=\{A, A^C\}, \mathcal{B} = \{B, B^C\}$. We denote

$$p_A = P(A), \quad p_{\overline{A}} = P(A^C), \quad p_B = P(B), \quad p_{\overline{B}} = P(B^C), \quad p_{AB} = P(A \cap B).$$

Calculation of $\mathrm{Var}(\mathbb{G}^{(N)}(f))$

For $N = 1, 2$ we have

$$\begin{aligned}
\mathrm{Var}(\mathbb{G}^{(1)}(f)) &= \mathrm{Var}(\mathbb{G}^{(0)}(f)) - \mathbb{E}[f|\mathcal{A}]^t \cdot \mathrm{Var}(\mathbb{G}[\mathcal{A}]) \cdot \mathbb{E}[f|\mathcal{A}] \\
&= \mathrm{Var}(\mathbb{G}^{(0)}(f)) - p_A p_{\overline{A}} (\mathbb{E}[f|A] - \mathbb{E}[f|A^C])^2, \\
\mathrm{Var}(\mathbb{G}^{(2)}(f)) &= \mathrm{Var}(\mathbb{G}^{(0)}(f)) - p_B p_{\overline{B}} (\mathbb{E}[f|B] - \mathbb{E}[f|B^C])^2 \\
&\quad - \left(p_A p_{\overline{A}} + \frac{p_B p_{\overline{B}} (p_{AB} - p_A p_B)}{p_A^2 p_{\overline{A}}^2} \right) (\mathbb{E}[f|A] - \mathbb{E}[f|A^C])^2,
\end{aligned}$$

Calculation of $\mathrm{Var}(\mathbb{G}^{(\infty)}(f))$

We denote $\Delta_A = \mathbb{E}[f|A] - \mathbb{E}[f], \Delta_B = \mathbb{E}[f|B] - \mathbb{E}[f]$, then

$$\mathrm{Var}(\mathbb{G}^{(\infty)}(f)) = \mathrm{Var}(\mathbb{G}^{(0)}(f)) - \frac{p_A p_B \left(p_A \Delta_A^2 + p_B \Delta_B^2 - p_A p_B (\Delta_A - \Delta_B)^2 - 2 p_{AB} \Delta_A \Delta_B\right)}{p_A p_B p_{\overline{A}} p_{\overline{B}} - (p_{AB} - p_A p_B)^2}.$$

In particular, if $\Delta_A = \Delta_B = 0$ then $\mathrm{Var}(\mathbb{G}^{(\infty)}(f)) = \mathrm{Var}(\mathbb{G}^{(0)}(f))$.

If A is independent of B then

$$\mathrm{Var}(\mathbb{G}^{(\infty)}(f)) = \mathrm{Var}(\mathbb{G}^{(0)}(f)) - \left(\frac{p_A}{p_{\overline{A}}}\Delta_A^2 + \frac{p_B}{p_{\overline{B}}}\Delta_B^2\right).$$

Extension 1: re-sampling method with auxiliary information

Notation

Bootstrap is a statistical method for re-sampling. It replaces P by $\mathbb{P}_n$.

A general way to define the bootstrap is to multiply $f(X_i)$ by a random variable Z_i such that $\mathbb{E}[Z_i|X_i] = 1$ and $\mathrm{Var}(Z_i) = 1$.

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We define the bootstrapped empirical measure and process:

$$\mathbb{P}_n^*(f) = \frac{1}{\sum_{i=1}^n Z_i} \sum_{i=1}^n Z_i f(X_i), \qquad \alpha_n^*(f) = \sqrt{n}(\mathbb{P}_n^*(f) - \mathbb{P}_n(f)).$$

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Goal

- Make the strong approximation of α_n^* to $\mathbb{G}^*$, a P-Brownian bridge independent of $\mathbb{G}$;
- Bootstrap the Raking-Ratio empirical process to simulate its distribution.

Strong approximation of the bootstrapped empirical process

Strong approximation of α_n^*

Under some entropy conditions on $\mathcal{F}$ we can construct on the same probability space (X_n, Z_n) and $(\mathbb{G}_n,, \mathbb{G}_n^*)$ of P-Brownian bridge such that for large n,

$$\mathbb{P}\left(\{||\alpha_n - \mathbb{G}_n||_{\mathcal{F}} > Cv_n\}\bigcup\{||\alpha_n^* - \mathbb{G}_n^*||_{\mathcal{F}} > Cv_n\}\right) \leqslant \frac{1}{n^2},$$

with $v_n \to 0$ depends on the entropy of $(\mathcal{F}, P)$.

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How can we adapt the bootstrap method to simulate the distribution of the Raking-Ratio empirical process?

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$$\alpha_n^{*(N)}(f) = \sqrt{n}(\mathbb{P}_n^{*(N)}(f) - \mathbb{P}_n(f)).$$

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$$\alpha_n^{*(N)}(f) = \sqrt{n}(\mathbb{P}_n^{*(N)}(f) - \mathbb{P}_n(f)).$$

Result

$\alpha_n^{*(N)} \to \mathbb{G}^{*(N)}$ in $\ell^\infty(\mathcal{F})$ and $\mathbb{G}^{*(N)}$ has the same distribution as $\mathbb{G}^{(N)}$.

Thank you for your attention!

Questions?