Travelling wave solutions for an evolutionary-epidemic problem arising in plant disease.

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Model with a spatial structure

Model the epidemiological and evolutionary dynamics of spore-producing pathogens in homogeneous host population of plants.

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Lambda - \mu u(t,x) - u(t,x) \displaystyle\int_{\mathbb{R}^M} \beta(y) w(t,x,y) dy, \\ \dfrac{\partial v(t,x,y)}{\partial t} = \beta(y) u(t,x) w(t,x,y) - \mu_v v(t,x,y), \\ \eta \dfrac{\partial w(t,x,y)}{\partial t} + \left(1 - \dfrac{\partial^2}{\partial x^2}\right) w(t,x,y) = \displaystyle\int_{\mathbb{R}^M} J\left(y - y'\right) r(y') v\left(t,x,y'\right) dy', \end{cases}$$

- $x \in \mathbb{R}$ (space), $y \in \mathbb{R}^M$ (phenotypic trait value), t (time).
- $u(t,x)$, $v(t,x,y)$ and $w(t,x,y)$ denote the density of healthy population, infected population and airborne spores, resp.
- $\Lambda > 0$ the influx of new healthy population, $\mu > 0$ natural death rate, $\eta > 0$ time scaling parameter and J mutation kernel.

Symmetric re-formulation of the model

We set $\eta = 0$ so that deposition of spores is fast with respect to the other processes.

Self-adjoint reformulation by setting $\Theta(y) = \sqrt{r(y)\beta(y)}$ and other parameter re-scaling lead

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Lambda - u(t,x) - u(t,x) \displaystyle\int_{\mathbb{R}^M} \beta(z) w(t,x,y) dy, \\ \dfrac{\partial v(t,x,y)}{\partial t} = u(t,x) w(t,x,y) - \mu_v v(t,x,y), \\ \left(1 - \dfrac{\partial^2}{\partial x^2}\right) w(t,x,y) = \displaystyle\int_{\mathbb{R}^M} \Theta(y)\Theta(y') J\left(y - y'\right) v\left(t,x,y'\right) dy', \end{cases}$$

$\Theta(y)$ denotes the fitness function, and we consider the operator L defined

by,

$$L[\varphi](y) = \Theta(y) \int_{\mathbb{R}^M} J(y-y')\Theta(y')\varphi(y')dy', \ y \in \mathbb{R}^M.$$

Assumptions

We assume :

a) The mutation kernel J is continuous and satisfies $J(-y) = J(y)$ for all $y \in \mathbb{R}^N$,

$$J \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ J > 0 \text{ and } \int_{\mathbb{R}^N} J(y)dy = 1.$$

a) The fitness function $\Theta : \mathbb{R}^N \to \mathbb{R}$ is non-negative, compactly supported and continuous. We denote by $\Omega \subset \mathbb{R}^M$ the open set defined by

$$\Omega = \left\{ y \in \mathbb{R}^N :\ \Theta(y) > 0 \right\}.$$

c) We also assume that $\beta : \mathbb{R}^N \to \mathbb{R}$ with $\beta \not\equiv 0$ is a non-negative continuous function with compact support and there exists some constant $K > 0$ such that

$$0 \leq \beta(y) \leq K\Theta(y),\ \forall y \in \mathbb{R}^N.$$

Epidemic threshold and equilibria

Under the above assumptions, $\varphi \mapsto \Theta J * (\Theta\varphi)$ is irreducible, positive and compact in $L^p(\Omega)$ (and self-adjoint in L^2) so that spectral decomposition $(\varphi_n, \lambda_n)_{n\geq 1}$ with

$$\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \to 0 \text{ as } n \to \infty \text{ and } \varphi_1 > 0.$$

We set $\mathcal{R}_0 = \frac{\lambda_1 \Lambda}{\mu_v}$, the epidemic threshold so that

(i) Single equilibrium when $\mathcal{R}_0 \leq 1$, the DFE ($U = \Lambda$),

(ii) Two equilibria when $\mathcal{R}_0 > 1$, the DFE and an endemic one,

$$(U, V, W) = \left(U^*, V^*\varphi_1(\cdot), W^*\varphi_1(\cdot)\right),$$

with

$$(U^*, V^*, W^*) = \left(\frac{\Lambda}{\mathcal{R}_0}, \frac{\mathcal{R}_0 - 1}{\lambda_1\beta_1}, \frac{\mathcal{R}_0 - 1}{\beta_1}\right) \text{ and } \beta_1 = \langle \beta, \varphi_1 \rangle_{L^2}.$$

Travelling wave

Definition :

$(u(t,x), v(t,x,y), w(t,x,y))$ is a TW with speed $c > 0$ if
$(u(t,x), v(t,x,y), w(t,x,y)) \equiv (U(\xi), V(\xi,y), W(\xi,y))$ with $\xi = x + ct$,
with the wave profile (U, V, W) satisfying the following properties :

- $U \in C^1 \cap L^\infty(\mathbb{R})$, $V, W : \mathbb{R} \to L^1(\mathbb{R}^N)$, C^1 and C^2 and

$$\sup_{\xi \in \mathbb{R}} \left[\|V(\xi,\cdot)\|_{L^1(\mathbb{R}^N)} + \|W(\xi,\cdot)\|_{L^1(\mathbb{R}^N)} \right] < \infty,$$

- $(U(\xi), V(\xi,y), W(\xi,y))$ positive.
- (U, V, W) satisfies the convergence to the DFE at $\xi = -\infty$

$$\lim_{\xi \to -\infty} \begin{pmatrix} U(\xi) \\ V(\xi,y) \\ W(\xi,y) \end{pmatrix} = \begin{pmatrix} \Lambda \\ 0 \\ 0 \end{pmatrix} \text{ in } \mathbb{R} \times L^1(\mathbb{R}^M) \times L^1(\mathbb{R}^M).$$

Wave profile system

A wave profile (U, V, W) with speed c satisfies

$$\begin{cases} c\dfrac{d}{d\xi}U(\xi) = \Lambda - U(\xi) - U(\xi)\displaystyle\int_\Omega \beta(z)W(\xi,y)dy, \\ c\dfrac{\partial}{\partial\xi}V(\xi,y) = U(\xi)W(\xi,y) - \mu_v V(\xi,y), \\ \left(1 - \dfrac{\partial^2}{\partial\xi^2}\right)W(\xi,y) = L\left[V(\xi,\cdot)\right](y), \end{cases}$$

$$\text{with } \lim_{\xi\to-\infty}\begin{pmatrix} U(\xi) \\ V(\xi,y) \\ W(\xi,y) \end{pmatrix} = \begin{pmatrix} \Lambda \\ 0 \\ 0 \end{pmatrix} \text{ in } \mathbb{R}\times L^1(\Omega)\times L^1(\Omega),$$

The minimal wave speed

The antsatz

$$V(\xi, y) \approx e^{\lambda\xi}\phi(y) \text{ and } W(\xi, y) \approx e^{\lambda\xi}\psi(y) \text{ as } \xi \to -\infty,$$

with the exponential decay rate $\lambda > 0$ while $U(\xi) \approx \Lambda$ for $\xi \ll -1$. Hence formally we get

$$\begin{cases} (c\lambda + \mu_v)\,\phi = \Lambda\psi, \\ (1-\lambda^2)\,\psi = L\phi, \end{cases} \quad \Leftrightarrow \quad \begin{cases} (c\lambda + \mu_v)\,\phi = \Lambda\psi, \\ \frac{1}{\Lambda}\,(1-\lambda^2)\,(c\lambda + \mu_v)\,\phi = L\phi. \end{cases}$$

$$\Rightarrow \frac{1}{\Lambda}\left(1-\lambda^2\right)(c\lambda + \mu_v) = \rho(L) = \lambda_1.$$

We define $\mathcal{K}(\lambda, c) = (1-\lambda^2)\,(c\lambda + \mu_v) - \mu_v\mathcal{R}_0$ and

$$c^\star = \inf\{c > 0,\ \exists\lambda > 0,\ \mathcal{K}(\lambda, c) = 0\}.$$

Non existence

We obtain the following non-existence result

(i) If $\mathcal{R}_0 \leq 1$ then the problem does not has any travelling wave solution.

(ii) If $\mathcal{R}_0 > 1$ then the problem does not admit any travelling wave solution with wave speed $c \in (0, c^\star)$.

Simple structure of the waves

Assume that $\mathcal{R}_0 > 1$. Let (U, V, W) be any TW with speed $c \geq c^\star$. Then one has

(i) $(V(\xi, y), W(\xi, y)) \equiv (V(\xi)\varphi_1(y), W(\xi)\varphi_1(y))$

(ii) The function $(U(\xi), V(\xi), W(\xi))$ satisfies the 4th order ODE

$$\begin{cases} cU'(\xi) = \Lambda - U(\xi) - \beta_1 U(\xi)W(\xi), \\ cV'(\xi) = U(\xi)W(\xi) - \mu_v V(\xi), & \xi \in \mathbb{R}, \\ \lambda_1 V(\xi) + \left(\frac{d^2}{d\xi^2} - 1\right) W(\xi) = 0, \end{cases}$$

and the limit behaviours at $\xi = \pm\infty$:

$$\lim_{\xi \to -\infty} \begin{pmatrix} U(\xi) \\ V(\xi) \\ W(\xi) \end{pmatrix} = \begin{pmatrix} \Lambda \\ 0 \\ 0 \end{pmatrix} \text{ and } \lim_{\xi \to \infty} \begin{pmatrix} U(\xi) \\ V(\xi) \\ W(\xi) \end{pmatrix} = \begin{pmatrix} U^* \\ V^* \\ W^* \end{pmatrix},$$

Reduction of the ∞-dim travelling wave profile system of equations to a 4-dim ODE system.

Existence of wave

Assume $\mathcal{R}_0 > 1$. Then for each $c \geq c^\star$, the above 4th order ODE has an entire positive solution with the expected behaviour at $\xi = \pm\infty$.

Idea of the proof :

The proof of these results is based on the projection of the travelling profile (V, W) on the eigenvectors (φ_n) of the linear operator L.

Reference : L. Abi Rizk et al., Travelling wave solutions for a non-local evolutionary-epidemic system, J. Differential Equations (2019), https ://doi.org/10.1016/j.jde.2019.02.012